

Operational Quasiprobabilities for Qudits

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We propose a quasiprobability function, which is operationally defined and commensurate with experimental circumstance in the sense that its functionals exactly coincide with what they are supposed to represent. This enables to *directly* compare quantum statistics to those of a local hidden variable model with noninvasive measurability imposed on, called a classical model. We prove that the commensurate quasiprobability admits negative value(s) for a quantum state, whereas it is nonnegative for the classical model, being a legitimate probability function. It is remarkable that “nonclassicality” of a given quantum state is *operationally* determined. In addition we derive a sufficient condition for the entanglement of qudits in terms of a marginal quasiprobability of spatial correlations.

INTRODUCTION

Quantum physics contains distinct features from classical physics such as complementarity and nonlocality. It has generally been acknowledged how difficult to understand quantum physics and to unravel its features as they (or some of them) are counter-intuitive. As a transparent methodology, however, the comparison of quantum and classical statistics has provided significant intuitions in understanding quantum physics and separating its features from the classical. For instance, photons have been shown to occasionally exhibit antibunching that no classical statistics of light can imply. Those features are said “nonclassical” in a sense that the *classical* theory of light does not predict.

To compare quantum with classical statistics, Wigner function has been employed on phase space, which corresponds to a probability distribution function in classical statistics [1, 2]. Due to complementary (or uncertainty) principle, however, the Wigner function is not always positive semidefinite and negative-valued for some quantum states. As it is not allowed by any classical distribution, the negativity is regarded as a signature of the nonclassicality. The Wigner function, called a quasiprobability (distribution) function, has been generalized to discrete systems particularly of qubits as quantum informatics attains its importance [3, 4]. The generalized quasiprobability has been applied to omnidirectional range of quantum information processing, including quantum tomography, quantum teleportation, and analysis of quantum algorithms [5, 6].

Even though they have established significant progress in their own context, the previous approaches are “incommensurate” with experimental circumstances in the sense that all functionals do not necessarily coincide with what they seem to stand for. For instance, consider a functional of continuous-variable Wigner function $W(x, p)$, $C[W] = \int dx dp xp W(x, p)$. One might regard this as the coincident or consecutive expectation, $\langle xp \rangle$ of position x and momentum p . This is *de facto* a different measurement whose basis is a set of the eigenvectors of a Hermitian observable operator $(\hat{x}\hat{p} + \hat{p}\hat{x})/2$ [2]; those eigenvectors have nothing to do with the coincident nor consecutive measurements of the observable operators \hat{x} and \hat{p} . In the sense, any tries to *directly* compare such an incommensurate quasiprobability to its classical counterpart are pointless or misleading. In fact, the previous approaches make it difficult to interpret “classicality” [7].

In this Letter, we propose an alternative approach to define a *commensurate* quasiprobability function, by which we can directly compare quantum statistics to the classical. For this purpose it is crucial to specify a classical model and as one we take a local hidden variable model with noninvasive measurability imposed on [8–10]. We prove that for the classical model every commensurate quasiprobability is positive semidefinite, being a legitimate probability function. Based on the proof, we classify classical and nonclassical states. It is remarkable that the nonclassicality is *operationally* determined, depending on a given experimental circumstance. In addition we derive a sufficient condition for the entanglement of two qudits in terms of a marginal quasiprobability.

COMMENSURATE QUASIPROBABILITY

Quantum mechanical description

Measurements are at the heart of quantum theory. Their description is also central in our defining a commensurate quasiprobability. Suppose that K possible (incompatible) observables A_k are *selectively* and *consecutively* measured

for a given system. Each *nondestructive* and *nondegenerate* measurement for A_k is configured at a distinct space-time point, $x_k = (t_k, \vec{r}_k)$ with $t_1 < t_2 < \dots < t_K$, if it is to be measured. Every local measurement for A_k is assumed to be represented by a basis set $\{|a\rangle_k\}$ for D possible values, $a \in \{0, 1, \dots, D-1\}$. For each run, we perform one of 2^K (consecutive) measurements, depending on the selection of the K observables, denoted by a tuple (n_1, n_2, \dots, n_K) ; A_k is selected if $n_k \neq 0$.

In quantum theory, we need to carefully describe consecutive measurements. For instance, suppose that two observables $A_{1,3}$ are selected to be measured. The local measurement for A_1 collapses a given initial quantum state $\hat{\rho}$ to one of eigenstates $|a_1\rangle_1$ in the probability $p(a_1) = {}_1\langle a_1|\hat{\rho}|a_1\rangle_1$. No local measurement is configured for A_2 . That for A_3 collapses now the state $|a_1\rangle_1$, previously collapsed into, to $|a_3\rangle_3$ in the probability $p(a_3|a_1) = {}_3\langle a_3|a_1\rangle_1^2$. This completes the given consecutive measurement. We employ a specific form of expectations, $\chi(n_1, 0, n_3, 0, \dots) = \sum_{a_1, a_3=0}^{D-1} \omega^{n_1 a_1 + n_3 a_3} p(a_3|a_1)p(a_1)$, where $\omega = e^{2\pi i/D}$ is a primitive D -th root of unity and $n_{1,3}$ are integers in the interval $[1, D)$. All expectations of such a form compose a function, which we call a characteristic function and denote by $\chi(\mathbf{n})$ with $\mathbf{n} = (n_1, n_2, \dots, n_K)$. When local measurements are not restricted to the von Neumann projective measures, we can employ positive operator valued measures [11]. In the case each local measurement for A_k is represented by a set of Kraus operators $\{\hat{A}_k(a)\}$ with an overcompleteness relation $\sum_{a=0}^{D-1} \hat{A}_k^\dagger(a)\hat{A}_k(a) = \mathbb{1}$, where $\mathbb{1}$ is an identity operator on the Hilbert space \mathcal{H}_d of dimension d . The characteristic function of a state $\hat{\rho}$ is given by

$$\chi(\mathbf{n}) = \text{Tr} \mathcal{T} \prod_{l=1}^K \left[\delta_{n_l, 0} \mathcal{I} + (1 - \delta_{n_l, 0}) \sum_{a=0}^{D-1} \omega^{n_l a} \mathcal{A}_l(a) \right] [\hat{\rho}], \quad (1)$$

where \mathcal{I} is an identity superoperator, $\mathcal{I}[\hat{\rho}] = \hat{\rho}$, and $\mathcal{A}_k(a)$ are Kraus superoperators for observable A_k , $\mathcal{A}_k(a)[\hat{\rho}] = \hat{A}_k(a)\hat{\rho}\hat{A}_k^\dagger(a)$. The product of two superoperators is defined by their composition, for instance $\mathcal{A}_2\mathcal{A}_1[\hat{\rho}] = \hat{A}_2\hat{A}_1\hat{\rho}\hat{A}_1^\dagger\hat{A}_2^\dagger$, and \mathcal{T} is a time ordering operator such that $\mathcal{T}\mathcal{A}_k\mathcal{A}_l = \mathcal{T}\mathcal{A}_l\mathcal{A}_k = \mathcal{A}_k\mathcal{A}_l$ for $t_k > t_l$. Note that Eq. (1) results from all the possible (consecutive) measurements.

We now define a commensurate quasiprobability function by a discrete Fourier transformation from $\chi(\mathbf{n})$:

$$w(\mathbf{a}) \equiv \frac{1}{D^K} \sum_{\mathbf{n}=0}^{D-1} \omega^{-\mathbf{a} \cdot \mathbf{n}} \chi(\mathbf{n}), \quad (2)$$

where $\mathbf{a} = (a_1, a_2, \dots, a_K)$, the sum over \mathbf{n} is the abbreviation of those over n_k for $k = 1, \dots, K$, and the inner product $\mathbf{a} \cdot \mathbf{n} = \sum_{k=1}^K a_k n_k$. By definition, $w(\mathbf{a})$ reproduces all the expectations $\chi(\mathbf{n})$ by the functionals,

$$\chi(\mathbf{n}) = \sum_{\mathbf{a}=0}^{D-1} \omega^{\mathbf{n} \cdot \mathbf{a}} w(\mathbf{a}), \quad (3)$$

where the sum over \mathbf{a} stands for those over all a_k . In other words, *every functional of $w(\mathbf{a})$ as in Eq. (3) coincides with what it is supposed to represent*, i.e., *the expectation on the right hand side of Eq. (1) that results faithfully according to the experimental circumstance*. The quasiprobability $w(\mathbf{a})$ is a real-valued function on the manifold of the K *incompatible* observables, which satisfies (Q.1) unit measure: $\sum_{\mathbf{a}} w(\mathbf{a}) = 1$, (Q.2) marginals: Summing $w(\mathbf{a})$ over a part of arguments results in a marginal quasiprobability of the rest, and (Q.3) the quantum probability for a single observable: The marginal quasiprobability of a single argument a_k , $w_k(a_k)$ is the quantum probability of the appropriate measurement for A_k , that is, $w_k(a_k) = {}_k\langle a_k|\hat{\rho}|a_k\rangle_k$. The importance of Q.2 and Q.3 has been emphasized for quantum tomography [4, 5]. The commensurate quasiprobability can be negative-valued like Wigner functions, whereas its positivity is ensured by a certain class of classical theories.

Local hidden variable theories with noninvasive measurability

As a classical model, we take a local hidden variable model with noninvasive measurability throughout this Letter. Various types of hidden variable models have widely been adopted in their own context [8–10, 12–15]. Nonlocality has been contrasted with local hidden variable models [8, 9, 13]. The local hidden variable models assume the existence of nonnegative joint probabilities involving all possible observations [9, 13]. Our classical model also begins with such joint probabilities, assuming the noninvasive measurability that the values of observables are not altered by past nor future (whatever) observations [10]. This is understood as a not only spatially local but also *temporally* local hidden variable model, that we call the classical model.

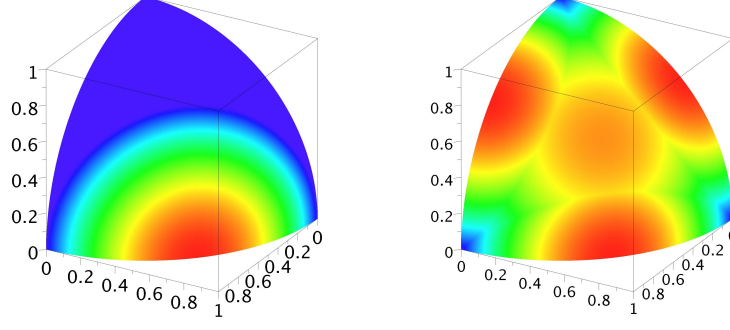


FIG. 1. Degrees of nonclassicality \mathcal{N} for pure states, corresponding to points on the Bloch sphere, for given observables (a) $\hat{A}_{1,2}$ such that $\hat{\sigma}_{x,y} = (-1)^{\hat{A}_{1,2}}$ and (b) $\hat{A}_{1,2,3}$ such that $\hat{\sigma}_{x,y,z} = (-1)^{\hat{A}_{1,2,3}}$. The degrees are converted into the hue color system that the dark blue color implies the lack of the nonclassicality. The maximal degrees $\mathcal{N}_{\max} = (\sqrt{2} - 1)/4$ is achieved if the qubit is in one of (a) four states $\vec{\rho} = (\pm 1, \pm 1, 0)/\sqrt{2}$ and (b) eight additional states $\vec{\rho} = (\pm 1, 0, \pm 1)/\sqrt{2}, (0, \pm 1, \pm 1)/\sqrt{2}$.

The assumptions of our classical model replace the quantum characteristic function in Eq. (1) by the classical,

$$\chi_{\text{cl}}(\mathbf{n}) = \sum_{\mathbf{a}=0}^{D-1} \omega^{\mathbf{n} \cdot \mathbf{a}} p_{\text{cl}}(\mathbf{a}), \quad (4)$$

where $p_{\text{cl}}(\mathbf{a})$ are (nonnegative) joint probabilities on the outcomes of all the possible observables, determined by the hidden variables (see the discussions around Eq. (2) in Ref. [9]). The quasiprobability function from $\chi_{\text{cl}}(\mathbf{n})$ by the Fourier transformation, as in Eq. (2), is exactly equal to the probability function: $w_{\text{cl}}(\mathbf{a}) = p_{\text{cl}}(\mathbf{a})$. This shows that *in our classical model every quasiprobability is nonnegative and it is a legitimate probability function.*

Classical and nonclassical states

Quantum theory conflicts with the classical model and it allows negative value(s) of a commensurate quasiprobability. If any, the negativity signifies “nonclassicality” in the sense that it can not be simulated by the classical model. A quantum state is said classical if its quasiprobability is nonnegative and otherwise nonclassical.

When the local measurements are of mutually unbiased bases (MUBs) (in which $D = d$) [16, 17], the quasiprobability function in Eq. (2) becomes [18]

$$w(\mathbf{a}) = \frac{1}{d^K} \left(1 + \sum_{k=1}^K \vec{\alpha}_k(a_k) \cdot \vec{\rho} \right), \quad (5)$$

where the generalized Bloch vectors $\vec{\rho} = \text{Tr} \vec{\lambda} \hat{\rho}$ and $\vec{\alpha}_k(a) = \text{Tr} \vec{\lambda} \hat{A}_k(a)$ with $\vec{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_{d^2-1})$. Here, $\{\hat{\lambda}_i\}$ is a complete orthogonal basis of Hermitian operators in Hilbert-Schmidt space of bound operators such that $\hat{\lambda}_0 = \mathbb{1}$ and $\text{Tr} \hat{\lambda}_i \hat{\lambda}_j = d \delta_{ij}$ for $i, j = 0, \dots, d^2 - 1$. Every Bloch vector $\vec{\rho}$ obeys $\vec{\rho} \cdot \vec{\tau} \geq -1$, for all Bloch vectors $\vec{\tau}$, due to the positivity of quantum states [17, 18].

To illustrate nonclassical states, we consider a simple system of a qubit ($d = 2$) on which two observables are measured ($K = 2$). The two observable operators $\hat{A}_{1,2}$ are assumed such that $\hat{\sigma}_x = (-1)^{\hat{A}_1}$ and $\hat{\sigma}_y = (-1)^{\hat{A}_2}$, where $\hat{\sigma}_{x,y,z}$ are Pauli spin operators. A quantum state is $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{\rho} \cdot \vec{\lambda})$, where the Bloch vector $|\vec{\rho}| \leq 1$ and $\vec{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. Then, the quasiprobability $w(a_1, a_2) = [1 + (-1)^{a_1} \rho_x + (-1)^{a_2} \rho_y]/4$. For a state of $\vec{\rho}$ orthogonal to the x - y plane by the observable operators, $w(a_1, a_2)$ is always positive semidefinite and the state is classical. On the other hand, a state of $\rho_x + \rho_y > 1$ is nonclassical as $w(1, 1) = (1 - \rho_x - \rho_y)/4 < 0$. In particular, if a state is pure and its $\vec{\rho}$ is on the x - y plane, one of $w(a_1, a_2)$ is always negative except $\vec{\rho} = \pm \hat{x}, \pm \hat{y}$. We quantify the nonclassicality by defining the degree \mathcal{N} as the absolute sum of all the negative values of $w(\mathbf{a})$: $\mathcal{N} = \frac{1}{2} \sum_{\mathbf{a}} (|w(\mathbf{a})| - w(\mathbf{a}))$. The nonclassicality degrees \mathcal{N} for pure states are presented in Fig. 1(a), where the states correspond to points on the Bloch sphere and their degrees are converted into the hue color system with dark blue

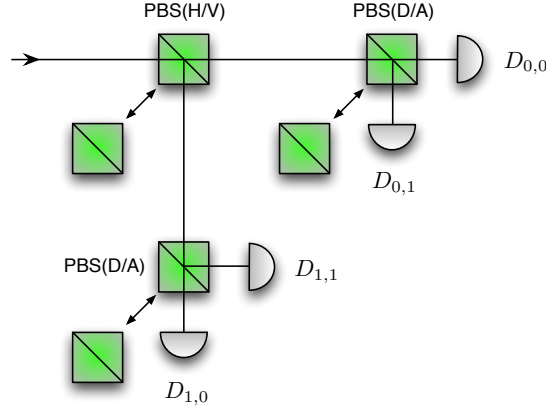


FIG. 2. A proposal for the optical experiment to construct quasiprobability function of a polarization qubit, where D_{a_1, a_2} are photon detectors and PBS are polarizing beam splitters of the horizontal/vertical (H/V) and diagonal/anti-diagonal (D/A). Each PBS is placed on the path or placed away, depending on the (consecutive) measurement (n_1, n_2) .

indicating $\mathcal{N} = 0$. Here, the maximal degree of nonclassicality is $\mathcal{N}_{\max} = (\sqrt{2}-1)/4$ for four states $\vec{\rho} = (\pm 1, \pm 1, 0)/\sqrt{2}$. In Fig. 1(b), the degrees are similarly presented when measuring three mutually complementary observables $\hat{A}_{1,2,3}$ such that $\hat{\sigma}_{x,y,z} = (-1)^{\hat{A}_{1,2,3}}$, in which case $K = 3$ and $w(a_1, a_2, a_3) = [1 + (-1)^{a_1}\rho_x + (-1)^{a_2}\rho_y + (-1)^{a_3}\rho_z]/8$. Now all the pure states are nonclassical except $\vec{\rho} = \pm\hat{x}, \pm\hat{y}, \pm\hat{z}$. The maximal nonclassicality is the same as the case of the two observables $\hat{A}_{1,2}$ and located at eight additional states $\vec{\rho} = (\pm 1, 0, \pm 1)/\sqrt{2}, (0, \pm 1, \pm 1)/\sqrt{2}$. It is interesting that the classical states for the three mutually complementary observables $\hat{A}_{1,2,3}$ coincide with those in Ref. [7], while they do not in other cases.

We would emphasize that *the nonclassicality or classicality of a quantum state is operationally determined in our approach of the commensurate quasiprobability* [17, 19]. For instance, in the circumstance of measuring just two observables $\hat{A}_{1,2}$, certain states are classical whereas they becomes nonclassical in that of measuring $\hat{A}_{1,2,3}$ (compare Figs. 1(a) and (b)).

OPTICAL SYSTEM WITH INEFFICIENT DETECTIONS

We propose an optical experiment of a polarization qubit to test if the classical or the quantum model properly predicts an experimental result. The experiment of measuring two observables is schematically presented in Fig. 2, where D_{a_1, a_2} are photon detectors and PBS is a polarizing beam splitter of the horizontal/vertical (H/V) or diagonal/anti-diagonal (D/A). Each PBS is placed on the path or placed away, depending on the (consecutive) measurement (n_1, n_2) of two observables, where $n_k = 0, 1$.

To obtain an experimental quasiprobability w_{exp} , letting $A_{1,2}$ be observables of H/V and D/A, we construct each expectation of (n_1, n_2) from the set of counts $\{N_{n_1 n_2}(a_1, a_2)\}$ at the detectors D_{a_1, a_2} and the total count number $N_{n_1 n_2}$: $\chi_{\text{exp}}(n_1, n_2) = \sum_{a_1, a_2} \omega^{n_1 a_1 + n_2 a_2} f_{n_1 n_2}(a_1, a_2)$, where $f_{n_1 n_2}(a_1, a_2) = N_{n_1 n_2}(a_1, a_2)/N_{n_1 n_2}$. Then, $w_{\text{exp}}(a_1, a_2)$ is given by the Fourier transformation as in Eq. (2).

SUFFICIENT CONDITION FOR ENTANGLEMENT

When applied to a composite system consisting of spatially separated subsystems, its commensurate quasiprobability reveals two types of nonclassicality. One is of the temporal correlation(s) of each subsystem which we discussed previously. The other is of the spatial correlations of the subsystems, that originates from entanglement. The segregation of the latter from the former can be made by considering a marginal quasiprobability of spatial correlations, which is nonnegative for any separable state whereas negative for an entangled state.

To illustrate, we consider two qudits 1 and 2 in a state $\hat{\rho}$ on the Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$ with d a prime or power of prime. Each qudit j is measured on $K (= d + 1)$ local observables A_{jk} , as for the single qudit. The quasiprobability $w(\mathbf{a})$ is a function of $\mathbf{a} = (a_{jk})$, where a_{jk} are the outcomes of A_{jk} . We take $w_c(\mathbf{c}) = \sum_{\mathbf{a}} (\prod_k \delta_{c_k, a_{1k} + a_{2k}}) w(\mathbf{a})$ as a marginal quasiprobability of spatial correlations. Here the Kronecker delta $\delta_{a,b} = 1$ if a and b are congruent modulo

d. The marginal quasiprobability is given [18] by

$$w_c(\mathbf{c}) = \frac{1}{d^{d+1}} (1 + \text{Tr} \mathbf{S} \mathbf{M}(\mathbf{c})), \quad (6)$$

where $\mathbf{S} = \text{Tr} \vec{\lambda} \otimes \vec{\lambda}^T \hat{\rho}$ and $\vec{\lambda}^T = (\hat{\lambda}_1^T, \dots, \hat{\lambda}_{d^2-1}^T)$ with the transposition T defined in the standard basis $\{|n\rangle\}$. Here,

$$\mathbf{M}(\mathbf{c}) = \frac{1}{d} \sum_k \sum_a \vec{\alpha}_{2,k}(c_k - a) \vec{\alpha}_{1,k}(a), \quad (7)$$

where the generalized Bloch vectors $\vec{\alpha}_{1k}(a) = \text{Tr} \vec{\lambda} \hat{A}_{1k}(a)$ and $\vec{\alpha}_{2k}(a) = \text{Tr} \vec{\lambda}^T \hat{A}_{2k}(a)$ for the two complete sets of MUB $\{\hat{A}_{jk}(a)\}$ of qudits $j = 1, 2$. For each \mathbf{c} , the matrix $\mathbf{M}(\mathbf{c})$ maps a generalized Bloch vector of qudit 1 to that of qudit 2, preserving the vector norm [18].

For an arbitrary separable state $\hat{\rho} = \sum_i p_i \hat{\rho}_{1i} \otimes \hat{\rho}_{2i}$, $\mathbf{S} = \sum_i p_i \vec{\rho}_{1i} \vec{\rho}_{2i}$ and Eq. (44) becomes

$$w_c(\mathbf{c}) = \frac{1}{d^{d+1}} \left(1 + \sum_i p_i \vec{\rho}_{2i} \cdot \mathbf{M}(\mathbf{c}) \cdot \vec{\rho}_{1i} \right). \quad (8)$$

Note that $\vec{\rho}_{2i} \cdot \mathbf{M}(\mathbf{c}) \cdot \vec{\rho}_{1i} \geq -1$ for each i , because $\mathbf{M} \cdot \vec{\rho}_{1i}$ is a Bloch vector of qudit 2 and every pair of Bloch vectors have an overlap no less than -1 due to the state positivity [17, 18]. Thus, the marginal quasiprobability is positive semidefinite for every separable state, whereas it can be negative for some entangled states [18]. This reciprocally implies *the sufficient condition that a quantum state is entangled if its marginal quasiprobability of spatial correlations has any negative value*. In the sense, the marginal play a role of entanglement witness [20].

SUMMARY

we proposed a commensurate quasiprobability function to faithfully describe (consecutive) measurements in the operational way. It was proved that it is positive semidefinite and a legitimate probability function for every local hidden variable model with the noninvasive measurability. We explicitly illustrated classical and nonclassical states for a qubit and showed that the nonclassicality is operationally determined, depending on give experimental circumstance. In analyzing the entanglement we proposed a marginal quasiprobability of spatial correlations to segregate spatial nonclassicality from temporal nonclassicality and showed that its negative value, if any, is a sufficient condition for two-qudit entanglement. It will be interesting to apply to quantum information processing, for instance, testing if any found algorithm is classically simulated, and to Bell's inequalities for an arbitrary system. It is an open question how to define a commensurate quasiprobability for a continuous variable system, which might suffer from theoretical difficulties when its observables are unbound [21].

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SUPPLEMENTARY INFORMATION

We derive explicitly the crucial results presented in the paper. For the purpose we employ the generalized Bloch representation and provide some properties in the notations of the paper. For a single qudit, we derive the form of commensurate quasiprobability function when local measurements are given by mutually unbiased bases. For two qudits, we derive a marginal quasiprobability of spatial correlations, in addition to the characteristic and quasiprobability functions. We prove the sufficient condition for two-qudit entanglement that a quantum state is entangled if its marginal quasiprobability of spatial correlations has any negative value. As a nontrivial example of applying the theorem, we classify one-parameter family of two-qudit Werner states.

Generalized Bloch representation

A generalized Bloch representation is convenient and employed to derive the crucial results in the paper. We summarize the results given by Ref. [17] and provide some additional properties in the notations of the paper.

Consider an orthogonal complete set of Hermitian operators, $\{\hat{\lambda}_i | i = 0, \dots, d^2 - 1\}$, on the Hilbert-Schmidt space of bound operators such that $\hat{\lambda}_0 = \mathbb{1}$ and $\text{Tr} \hat{\lambda}_i \hat{\lambda}_j = d \delta_{i,j}$ where $\delta_{i,j}$ is a Kronecker delta. By using the set, a Hermitian operator \hat{H} and a density operator $\hat{\rho}$ are expanded to

$$\hat{H} = \frac{1}{d} \left(h_0 \hat{\lambda}_0 + \sum_{\alpha=1}^{d^2-1} h_\alpha \hat{\lambda}_\alpha \right) \quad (9)$$

$$\hat{\rho} = \frac{1}{d} \left(\rho_0 \hat{\lambda}_0 + \sum_{\alpha=1}^{d^2-1} \rho_\alpha \hat{\lambda}_\alpha \right), \quad (10)$$

where $h_i = \text{Tr} \hat{\lambda}_i \hat{H}$ and $\rho_i = \text{Tr} \hat{\lambda}_i \hat{\rho}$. For every density operator (unit-trace positive operator) $\hat{\rho}$, $\rho_0 = 1$ and we call $\vec{\rho} = (\rho_1, \dots, \rho_{d^2-1})$ a generalized Bloch vector (GBV). We employ a concise notation hereafter:

$$\vec{\rho} = \text{Tr} \vec{\lambda} \hat{\rho}, \quad (11)$$

where $\vec{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_{d^2-1})$. Because $\text{Tr} \hat{\rho}^2 \leq 1$, the norm of $\vec{\rho}$ is upper bounded: $|\vec{\rho}| \leq \sqrt{d-1}$. If $\hat{\rho}$ is pure, $|\vec{\rho}| = \sqrt{d-1}$. The GBVs stay within a Bloch sphere of radius $\sqrt{d-1}$. However, not all vectors within the Bloch sphere correspond to density operators, that is, there is no one-to-one correspondence between density operators and vectors within the Bloch sphere. In fact, the set of GBVs specifying density operators is restricted by the positivity of density operators such that a given density operator $\hat{\rho}$ should hold

$$\text{Tr} \hat{\rho} \hat{\tau} \geq 0 \Leftrightarrow \vec{\rho} \cdot \vec{\tau} \geq -1, \quad (12)$$

for all pure density operators $\hat{\tau}$ with $|\vec{\tau}| = \sqrt{d-1}$.

A density operator for two qudits, $\hat{\rho}$ is given in the Bloch representation by

$$\hat{\rho} = \frac{1}{d^2} \left(\mathbb{1} \otimes \mathbb{1} + \vec{\rho}_1 \cdot \vec{\lambda} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{\rho}_2 \cdot \vec{\lambda}^T + \sum_{\alpha, \beta=1}^{d^2-1} S_{\alpha\beta} \hat{\lambda}_\alpha \otimes \hat{\lambda}_\beta^T \right), \quad (13)$$

where $\vec{\rho}_j$ are local GBVs of qudit j and $\mathbf{S} = \text{Tr} \vec{\lambda} \otimes \vec{\lambda}^T \hat{\rho}$. Here $\vec{\lambda}^T = (\hat{\lambda}_1^T, \hat{\lambda}_2^T, \dots, \hat{\lambda}_{d^2-1}^T)$ and the operator transposition $\hat{\lambda}_\alpha^T$ is defined in the standard basis $\{|n\rangle\}$. We call \mathbf{S} a generalized Bloch matrix (GBM). From the upper bound $\text{Tr} \hat{\rho}^2 \leq 1$, the local GBVs and GBM obey,

$$\sum_{j=1,2} |\vec{\rho}_j|^2 + \text{Tr} \mathbf{S}^T \mathbf{S} \leq d^2 - 1. \quad (14)$$

The positivity of $\hat{\rho}$ is represented by

$$\sum_{j=1,2} \vec{\rho}_j \cdot \vec{\tau}_j + \text{Tr} \mathbf{S} \mathbf{T}^T \geq -1, \quad (15)$$

where $\vec{\tau}_j$ are the local GBVs and \mathbf{T} is the GBM of an arbitrary pure state $\hat{\tau}$. If the local GBVs $\vec{\rho}_1 = \vec{\rho}_2 = 0$ such as maximally entangled states and Werner states, the inequality (14) becomes

$$\text{Tr } \mathbf{S}^T \mathbf{S} \leq d^2 - 1 \quad (16)$$

and the inequality (15) of state positivity becomes

$$\text{Tr } \mathbf{S} \mathbf{T}^T \geq -1. \quad (17)$$

In particular, if we are focused on any pure product state $\hat{\tau} = \hat{\tau}_1 \otimes \hat{\tau}_2$, then $\mathbf{T} = \vec{\tau}_1 \vec{\tau}_2$ so that the state positivity implies

$$\vec{\tau}_1 \cdot \mathbf{S} \cdot \vec{\tau}_2 \geq -1. \quad (18)$$

Some properties of GBVs and GBMs are summarized.

(G.1) A vector $\vec{\rho}$ is a GBV if and only if satisfying the two conditions;

(D.1) being within the Bloch sphere, $|\vec{\rho}| \leq \sqrt{d-1}$, and

(D.2) the state positivity as in Eq. (12), $\vec{\rho} \cdot \vec{\tau} \geq -1$ for every pure GBV $\vec{\tau}$ with $|\vec{\tau}| = \sqrt{d-1}$.

(G.2) Given a GBV $\vec{\rho}$, the vector $\vec{\sigma} = \xi \vec{\rho}$ is a GBV for $0 \leq \xi \leq 1$:

It is clear that $\vec{\sigma}$ satisfies the condition (D.1). It also does (D.2) as

$$\vec{\sigma} \cdot \vec{\tau} = \xi \vec{\rho} \cdot \vec{\tau} \geq -\xi \geq -1, \quad (19)$$

for every pure GBV $\vec{\tau}$.

(G.3) A convex combination of GBVs is a GBV:

Let $\vec{\rho}_i$ be GBVs and $0 \leq \xi_i \leq 1$ with $\sum_i \xi_i = 1$. The vector $\vec{\rho} = \sum_i \xi_i \vec{\rho}_i$ satisfies (D.1) as its norm is no larger than the largest among the GBVs $\vec{\rho}_i$ and thus $|\vec{\rho}|^2 \leq d-1$. The vector $\vec{\rho}$ also satisfies (D.2): For every GBV $\vec{\tau}$,

$$\vec{\rho} \cdot \vec{\tau} = \sum_i \xi_i \vec{\rho}_i \cdot \vec{\tau} \geq -\sum_i \xi_i = -1, \quad (20)$$

where we used $\vec{\rho}_i \cdot \vec{\tau} \geq -1$.

(D.3) A function from the set of GBVs to itself is called a Bloch map.

(G.4) For a two-qudit state $\hat{\rho}$ with local GBVs $\vec{\rho}_j = 0$, its GBM $\mathbf{S} = \text{Tr } \vec{\lambda} \otimes \vec{\lambda}^T \hat{\rho}$ satisfies $\vec{\tau}_1 \cdot \mathbf{S} \cdot \vec{\tau}_2 \geq -1$ for an arbitrary pair of GBVs $\vec{\tau}_j$, as shown in Eq. (18).

(G.5) An orthogonal GBM \mathbf{S} of a two-qudit state $\hat{\rho}$ is a Bloch map from qudit 2 to qudit 1:

For each GBV $\vec{\tau}$, the image $\vec{\sigma} = \mathbf{S} \cdot \vec{\tau}$ satisfies the condition (D.1) as \mathbf{S} is orthogonal. As $\text{Tr } \mathbf{S}^T \mathbf{S} = d^2 - 1$, by Eq. (14) all local GBVs of $\hat{\rho}$ vanish. By (G.4), the image $\vec{\sigma}$ satisfies (D.2). Thus \mathbf{S} is a Bloch map by the definition (D.3).

A nondegenerate and orthogonal measurement A is represented by a set of basis projectors $\{\hat{A}(a) = |a\rangle\langle a|\}$. The basis projectors are unit-trace positive operators, satisfying the orthonormality relation

$$\delta_{a,a'} = \text{Tr } \hat{A}(a) \hat{A}(a') = \frac{1}{d} (1 + \vec{\alpha}(a) \cdot \vec{\alpha}(a')), \quad (21)$$

where $\vec{\alpha}(a)$ are GBVs of $\hat{A}(a)$, $\vec{\alpha}(a) = \text{Tr } \vec{\lambda} \hat{A}(a)$. Eq. (21) is rewritten as $\vec{\alpha}(a) \cdot \vec{\alpha}(a') = d\delta_{a,a'} - 1$. The basis projectors also satisfy the completeness relation, given as

$$\sum_{a=0}^{d-1} \hat{A}(a) = \mathbb{1} \Leftrightarrow \sum_{a=0}^{d-1} \vec{\alpha}(a) = 0. \quad (22)$$

Their GBVs for the measurement A , $\vec{\alpha}(a)$ form an overcompleteness relation in the $(d-1)$ -dimensional subspace of Bloch sphere:

$$\frac{1}{d} \sum_{a=0}^{d-1} \vec{\alpha}(a) \vec{\alpha}(a) = \mathbb{1}_{d-1}, \quad (23)$$

where $\vec{\alpha}(a)\vec{\alpha}(a')$ is a tensor product and $\mathbb{1}_{d-1}$ is an identity matrix on the $(d-1)$ -dimensional subspace.

For a qubit with $d = 2$, a nondegenerate and orthogonal measurement A is given by two projectors, say $\hat{A}(a) = |a\rangle\langle a|$ for $a = 0, 1$. In the Bloch representation they are given as

$$\hat{A}(a) = \frac{1}{2} \left(\mathbb{1} + (-1)^a \vec{\alpha} \cdot \vec{\lambda} \right), \quad (24)$$

where $\vec{\alpha}$ is a unit vector of “orientation” for A and $\vec{\lambda} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli spin operators, $\hat{\sigma}_{x,y,z}$. The GBVs of projectors $\hat{A}(a)$ are thus $\vec{\alpha}(a) = (-1)^a \vec{\alpha}$.

Two measurements A_1 and A_2 are mutually complementary if precise knowledge in one of them implies that all possible outcomes in the other are equally probable [22, 23],

$$\text{Tr } \hat{A}_1(a) \hat{A}_2(b) = \frac{1}{d}, \quad \forall a, b = 0, \dots, d-1 \quad (25)$$

where $\{\hat{A}_k(a) = |a\rangle_k \langle a|\}$ is the set of basis projectors for A_k . This leads to the orthogonality relations of

$$\vec{\alpha}_1(a_1) \cdot \vec{\alpha}_2(a_2) = 0, \quad \forall a_1, a_2 = 0, \dots, d-1, \quad (26)$$

where $\vec{\alpha}_k(a)$ are GBVs of the basis projectors $\hat{A}_k(a)$ for A_k . It is then notable that the two mutually complementary measurements (MCM) have two exclusive Bloch subspaces, each of $(d-1)$ dimension. As the Bloch space is (d^2-1) -dimensional, there exist at most $(d+1)$ MCM. The existence of $(d+1)$ MCM was proved for prime or power of prime dimensions [16]. Even though the proof does not exclude the possibility of other dimensions, we assume that our subsystems are prime or power-of-prime dimensional to avoid any unnecessary confusion. The set of all MCM is said *complete*. The complete set of MCM allows the overcompleteness relation in the Bloch sphere,

$$\frac{1}{d} \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \vec{\alpha}_k(a) \vec{\alpha}_k(a) = \sum_{k=1}^{d+1} \mathbb{1}_{k,d-1} = \mathbb{1}_{d^2-1}, \quad (27)$$

where $\mathbb{1}_{k,d-1}$ is an identity matrix on the k -th exclusive Bloch subspace and $\mathbb{1}_{d^2-1}$ is that on the Bloch sphere. The basis sets of MCM are said mutually unbiased bases (MUB).

We add another property of GBVs related with MUB.

(G.6) For the given set $\{\vec{\alpha}_k(a)\}$, the two operator sets of $\{\hat{\lambda}_\alpha\}$ and $\{\hat{A}_k(a)\}$ are related by

$$\vec{\lambda} = \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \vec{\alpha}_k(a) \hat{A}_k(a) \quad (28)$$

$$\hat{A}_k(a) = \frac{1}{d} \left(\mathbb{1} + \vec{\alpha}_k(a) \cdot \vec{\lambda} \right). \quad (29)$$

The first transformation results from Eqs. (22) and (27), and the second does from Eqs. (26), (21), and (22).

(G.7) Quantum tomographic equivalence:

Consider a complete set of MUB, $\{\hat{A}_k(a)\}$ with the set of GBVs $\{\vec{\alpha}_k(a) = \text{Tr } \vec{\lambda} \hat{A}_k(a)\}$. The complete set expands a quantum state $\hat{\rho}$ as

$$\hat{\rho} = \frac{1}{d} \left(\mathbb{1} + \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \rho_k(a) \hat{A}_k(a) \right) \quad (30)$$

$$= \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} p_k(a) \hat{A}_k(a) - \mathbb{1}, \quad (31)$$

where we used the transformation as in Eq. (28) and $\rho_k(a) = \vec{\rho} \cdot \vec{\alpha}_k(a) = dp_k(a) - 1$ with $p_k(a) = \text{Tr } \hat{A}_k(a) \hat{\rho}$. Eq. (31) shows that every quantum state is completely determined by the complete set of MUB $\{\hat{A}_k(a)\}$ and their probability distributions $p_k(a)$ (or the coefficients $\rho_k(a)$). Consider another state, defined as

$$\hat{\sigma} = \frac{1}{d} \left(\mathbb{1} + \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \rho_k(a) \hat{B}_k(a) \right) \quad (32)$$

$$= \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} p_k(a) \hat{B}_k(a) - \mathbb{1}. \quad (33)$$

where the set $\{\rho_k(a) = dp_k(a) - 1\}$ is the same as that of $\hat{\rho}$ and $\{\hat{B}_k(a)\}$ is another complete set of MUB. We say that $\hat{\sigma}$ is equivalent to $\hat{\rho}$ in the sense that both are in the same form with respect to their complete sets of MUB and in particular both have the common probability distributions $p_k(a) = \text{Tr } \hat{B}_k(a)\hat{\sigma} = \text{Tr } \hat{A}_k(a)\hat{\rho}$. The two operators $\hat{\sigma}$ and $\hat{\rho}$ are connected by the relation,

$$\vec{\sigma} = \frac{1}{d} \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \vec{\beta}_k(a) \vec{\alpha}_k(a) \cdot \vec{\rho}, \quad (34)$$

where $\vec{\beta}_k(a) = \text{Tr } \vec{\lambda} \hat{B}_k(a)$. This implies that the matrix $\mathbf{M} = \frac{1}{d} \sum_k \sum_a \vec{\beta}_k(a) \vec{\alpha}_k(a)$ is a Bloch map from one complete set of MUB $\{\alpha_k(a)\}$ to another $\{\beta_k(a)\}$ (see (D.3)). As an example, consider a state $\hat{\rho}$ which is diagonal in the basis $\{\hat{A}_1(a)\}$: $\hat{\rho} = \sum_a p_1(a) \hat{A}_1(a)$. This is equal to Eq. (31) with $p_k(a) = 1/d$ for all pairs of a and $k \neq 1$. The operator $\hat{\sigma} = \sum_a p_1(a) \hat{B}_1(a)$ is a quantum state and $\vec{\sigma} = \frac{1}{d} \sum_a \vec{\beta}_1(a) \vec{\sigma}_1(a) \cdot \vec{\rho}$ is a GBV.

Quasiprobability for a qudit

We shall derive the form of commensurate quasiprobability function $w(\mathbf{a})$ when local measurements are of mutually unbiased bases (MUB). The quasiprobability function in Eq. (2) of the paper is rewritten as

$$w(\mathbf{a}) = \text{Tr} \left[\mathcal{T} \prod_{k=1}^K \left(\frac{\mathcal{I}}{D} + \Delta \mathcal{A}_k(a_k) \right) [\hat{\rho}] \right], \quad (35)$$

where $\mathbf{a} = (a_1, a_2, \dots, a_K)$, $\Delta \mathcal{A}_k(a) = \mathcal{A}_k(a) - \frac{1}{D} \sum_{a'} \mathcal{A}_k(a')$ for each k , and $\hat{\rho}$ is a quantum state. Here \mathcal{I} is an identity superoperator, $\mathcal{I}[\hat{\rho}] = \hat{\rho}$ and $\mathcal{A}_k(a)$ are Kraus superoperators of measurement A_k , $\mathcal{A}_k(a)[\hat{\rho}] = \hat{A}_k(a) \hat{\rho} \hat{A}_k^\dagger(a)$. The product of two superoperators is defined by their composition, for instance $\mathcal{A}_2 \mathcal{A}_1[\hat{\rho}] = \hat{A}_2 \hat{A}_1 \hat{\rho} \hat{A}_1^\dagger \hat{A}_2^\dagger$, and \mathcal{T} is a (time) ordering operator such that $\mathcal{T} \mathcal{A}_k \mathcal{A}_l = \mathcal{T} \mathcal{A}_l \mathcal{A}_k = \mathcal{A}_k \mathcal{A}_l$ for $k > l$.

If A_k are of MUB, they satisfy mutual complementarity in Eq. (25), each with $D = d$ outcomes, and so two *different* measurements A_k and A_l satisfy

$$\text{Tr } \mathcal{A}_k(a) \mathcal{A}_l(a') [\hat{\rho}] = \frac{1}{d} \text{Tr } \mathcal{A}_l(a') [\hat{\rho}], \quad (36)$$

for each a and a' . This leads to $\text{Tr } \Delta \mathcal{A}_k(a) \Delta \mathcal{A}_l(a') [\hat{\rho}] = 0, \forall \hat{\rho}$. The generalization is straightforward to arbitrary combinations of different measurements of MUB: For each (a_k, \dots, a_l) ,

$$\text{Tr } \Delta \mathcal{A}_k(a_k) \cdots \Delta \mathcal{A}_l(a_l) [\hat{\rho}] = 0, \forall \hat{\rho}. \quad (37)$$

In the case, the quasiprobability function is given as

$$w(\mathbf{a}) = \frac{1}{d^K} \text{Tr} \left[\left(\mathcal{I} + d \sum_{k=1}^K \Delta \mathcal{A}_k(a_k) \right) [\hat{\rho}] \right] = \frac{1}{d^K} \left(1 + \sum_{k=1}^K \vec{\alpha}_k(a_k) \cdot \vec{\rho} \right), \quad (38)$$

where $\vec{\rho}$ is the GBV of state $\hat{\rho}$ and $\vec{\alpha}_k(a)$ are those of projectors $\hat{A}_k(a)$. Eq. (38) is the result presented in the paper. For a qubit, Eq. (38) is reduced to

$$w(\mathbf{a}) = \frac{1}{2^K} \left(1 + \sum_{k=1}^K (-1)^{a_k} \vec{\alpha}_k \cdot \vec{\rho} \right), \quad (39)$$

where $\vec{\alpha}_k$ is the unit vector of the orientation A_k (see Eq. (24)).

Quasiprobability of two qudits

Consider a composite system consisting of two spatially separated qudits in a quantum state $\hat{\rho}$. Each qudit is to be selectively and consecutively measured by a complete set of $K = d + 1$ MUB with $D = d$, as described in the paper

for a single qudit. The two complete sets of MUB are not necessarily the same. Its characteristic and quasiprobability functions are respectively given, similarly to Eqs. (1) and (2) in the paper, as

$$\chi(\mathbf{n}) = \text{Tr} \bigotimes_{j=1}^2 \mathcal{T} \prod_{k=1}^{d+1} \left[\delta_{n_{jk},0} \mathcal{I} + (1 - \delta_{n_{jk},0}) \sum_{a=0}^{d-1} \omega^{n_{jk}a} \mathcal{A}_{jk}(a) \right] [\hat{\rho}] \quad (40)$$

and

$$w(\mathbf{a}) = \text{Tr} \bigotimes_{j=1}^2 \mathcal{T} \prod_{k=1}^{d+1} \left(\frac{\mathcal{I}}{d} + \Delta \mathcal{A}_{jk}(a_{jk}) \right) [\hat{\rho}], \quad (41)$$

where $\mathbf{a} = (a_{jk})$ with its elements being the value a_{jk} of k -th observable A_{jk} for qudit j and $\Delta \mathcal{A}_{jk}(a) = \mathcal{A}_{jk}(a) - \frac{1}{d} \sum_{a'} \mathcal{A}_{jk}(a')$ for each (j, k) (the other symbols are defined similarly to Eqs. (1) and (2) in the paper). By the similar reasoning to Eqs. (37) and (38), the quasiprobability results in

$$w(\mathbf{a}) = \frac{1}{d^{2(d+1)}} \left(1 + \sum_{k=1}^{d+1} \vec{\alpha}_{1k}(a_{1k}) \cdot \vec{\rho}_1 + \sum_{l=1}^{d+1} \vec{\alpha}_{2l}(a_{2l}) \cdot \vec{\rho}_2 + \sum_{k,l=1}^{d+1} \vec{\alpha}_{1k}(a_{1k}) \cdot \mathbf{S} \cdot \vec{\alpha}_{2l}(a_{2l}) \right), \quad (42)$$

where the local GBVs $\vec{\rho}_1 = \text{Tr} \vec{\lambda} \otimes \mathbb{1} \hat{\rho}$ and $\vec{\rho}_2 = \text{Tr} \mathbb{1} \otimes \vec{\lambda}^T \hat{\rho}$, the GBM $\mathbf{S} = \text{Tr} \vec{\lambda} \otimes \vec{\lambda}^T \hat{\rho}$, $\vec{\alpha}_{1,k}(a) = \text{Tr} \vec{\lambda} \hat{A}_{1,k}(a)$, and $\vec{\alpha}_{2,k}(a) = \text{Tr} \vec{\lambda}^T \hat{A}_{2,k}(a)$.

Marginal quasiprobability of spatial correlations and two-qudit entanglement

We define a marginal quasiprobability of spatial correlations by

$$w_c(\mathbf{c}) = \sum_{\mathbf{a}=0}^{d-1} \left(\prod_{k=1}^{d+1} \delta_{c_k, a_{1k} + a_{2k}} \right) w(\mathbf{a}), \quad (43)$$

where $\mathbf{c} = (c_1, c_2, \dots, c_{d+1})$ and the Kronecker delta $\delta_{a,b} = 1$ if a and b are congruent modulo d and $\delta_{a,b} = 0$ otherwise. Here the sum over \mathbf{a} is the abbreviation of those over a_{jk} for $j = 1, 2$ and $k = 1, 2, \dots, d+1$. By the relation for $\vec{\alpha}_{jk}(a)$ as in Eq. (22), the marginal becomes

$$w_c(\mathbf{c}) = \frac{1}{d^{d+1}} (1 + \text{Tr} \mathbf{S} \mathbf{M}(\mathbf{c})), \quad (44)$$

where the trace is defined on the Bloch sphere and

$$\mathbf{M}(\mathbf{c}) = \frac{1}{d} \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \vec{\alpha}_{2,k}(c_k - a) \vec{\alpha}_{1,k}(a). \quad (45)$$

Here, we note that the argument $c_k - a$ is defined by its positive residue modulo d .

The matrix $\mathbf{M}(\mathbf{c})$ for each \mathbf{c} is a Bloch map from qudit 2 to qudit 1, in other words, it maps a GBV of qudit 2 to qudit 1 (see (D.3), (G.5), and (G.7)). The matrix $\mathbf{M}(\mathbf{c})$ is an isometry (or orthogonal), preserving the vector norm: Letting $\mathbf{M} = \mathbf{M}(\mathbf{c})$ for a given \mathbf{c} ,

$$\begin{aligned} \mathbf{M}^T \mathbf{M} &= \frac{1}{d^2} \sum_{k,l=1}^{d+1} \sum_{a,b=0}^{d-1} \vec{\alpha}_{1,k}(a) \vec{\alpha}_{2,k}(c_k - a) \cdot \vec{\alpha}_{2,l}(c_l - b) \vec{\alpha}_{1,l}(b) \\ &= \frac{1}{d} \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \vec{\alpha}_{1,k}(a) \vec{\alpha}_{1,k}(a) \\ &= \mathbb{1}_{1,d^2-1}, \end{aligned} \quad (46)$$

where we used the relations as in Eqs. (21), (22), (26), and (27). Here, $\mathbb{1}_{j,d^2-1}$ is an identity matrix on the Bloch space of qudit j . Similarly,

$$\mathbf{M} \mathbf{M}^T = \frac{1}{d} \sum_{k=1}^{d+1} \sum_{a=0}^{d-1} \vec{\alpha}_{2,k}(c_k - a) \vec{\alpha}_{2,k}(c_k - a) = \mathbb{1}_{2,d^2-1}. \quad (47)$$

These properties of $\mathbf{M}(\mathbf{c})$ are employed in the paper to prove the sufficient condition that a quantum state is entangled if its marginal quasiprobability of spatial correlations $w_c(\mathbf{c})$ has any negative value.

The marginal function $w_c(\mathbf{c})$ in Eq. (44) plays a role of entanglement witness [20]. As a nontrivial example, we consider a Werner state of two qudits, $\hat{\rho}_w = f|\psi\rangle\langle\psi| + (1-f)\mathbb{1} \otimes \mathbb{1}/d^2$, where $|\psi\rangle = \sum_n |n, n\rangle/\sqrt{d}$ is a maximally entangled state and $0 \leq f \leq 1$. In the Bloch representation, it becomes

$$\hat{\rho}_w = \frac{1}{d^2} \left(\mathbb{1} \otimes \mathbb{1} + f \sum_{\alpha\beta=1}^{d^2-1} S_{\alpha\beta} \hat{\lambda}_\alpha \otimes \hat{\lambda}_\beta^T \right), \quad (48)$$

where $\mathbf{S} = \langle\psi|\vec{\lambda} \otimes \vec{\lambda}^T|\psi\rangle = \delta_{\alpha\beta}$ is the GBM of $|\psi\rangle$. Note that the GBM \mathbf{S} is orthogonal and it is a Bloch map from qudit 2 to qudit 1 so that for $\vec{\rho}_2$ a GBV of qudit 2, $\vec{\rho}_1 = \mathbf{S} \cdot \vec{\rho}_2$ is a GBV of qudit 1 (see (G.5)). The marginal function $w_c(\mathbf{c})$ is given from Eq. (44) as

$$w_c(\mathbf{c}) = \frac{1}{d^{d+1}} (1 + f \text{Tr} \mathbf{S} \mathbf{M}(\mathbf{c})). \quad (49)$$

This implies

$$\text{Tr} \mathbf{S} \mathbf{M}(\mathbf{c}) = \frac{1}{d} \sum_k \sum_a \vec{\alpha}_{1k}(a) \cdot \mathbf{S} \cdot \vec{\alpha}_{2k}(c_k - a) \geq -\frac{1}{d} \sum_k \sum_a = -(d+1), \quad (50)$$

where we used (D.2). The marginal function is thus lower bounded,

$$w_c(\mathbf{c}) \geq \frac{1}{d^{d+1}} [1 - (d+1)f]. \quad (51)$$

The equality in Eq. (51) holds for $\mathbf{c} = (0, 0, \dots, 0)$ if the complete set of MUB for qudit 2 is chosen such that $\vec{\alpha}_{2k}(a) = \mathbf{S}^T \cdot \vec{\alpha}_{1k}(c_k - a + 1)$ for each (k, a) . Then, the minimum is negative as far as $f > 1/(d+1)$, implying that the Werner state $\hat{\rho}_w$ in Eq. (48) is entangled if $f > 1/(d+1)$.